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Asymptotics and bounds for the zeros of Laguerre polynomials: a survey

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Abstract

Some of the work on the construction of inequalities and asymptotic approximations for the zeros $\lambda_{n,k}^{(\alpha)}$, $k = 1,2,\ldots,n$, of the Laguerre polynomial $L_n^{(\alpha)}(x)$ as $v = 4n + 2\alpha + 2 \to \infty$, is reviewed and discussed. The cases when one or both parameters n and α unrestrictedly diverge are considered. Two new uniform asymptotic representations are presented: the first involves the positive zeros of the Bessel function $J_\alpha(x)$, and the second is in terms of the zeros of the Airy function $\mathrm{Ai}(x)$. They hold for $k = 1,2,\ldots,[qn]$ and for $k=[pn],[pn]+1,\ldots,n$, respectively, where p and q are fixed numbers in the interval $(0,1)$. Numerical results and comparisons are provided which favorably justify the consideration of the new approximations formulas.

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1. Introduction

Let $\lambda_{n,k}^{(\alpha)}$, $k=1,2,\ldots$, denote the zeros of the Laguerre polynomial $L_n^{(\alpha)}(x)$, $\alpha > -1$, in increasing order. It is well-known that these zeros lie in the oscillatory region $0 < x < v$, where

$$v = 4n + 2\alpha + 2 \tag{1.1}$$

is the turning point of the differential equation satisfied by $L_n^{(\alpha)}(x)$. Throughout this paper we shall assume $\alpha > -1$ and v will be defined by (1.1). We shall refer to the differential

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equation

$$\frac{d^2y}{dt^2}+\left[\frac{v^2}{4}\left(\frac{1}{t}-1\right)+\frac{1-\alpha^2}{4t^2}\right]y=0, \tag{1.2}$$

which is satisfied by

$$y(t)=e^{-vt/2}(vt)^{(\alpha+1)/2}L_n^{(\alpha)}(vt). \tag{1.3}$$

Further we shall denote with $j_{\alpha,k}$, $k=1,2,\ldots$, the positive zeros, in increasing order, of the Bessel function $J_\alpha(x)$, or, more precisely by setting $j_{\alpha,0}$ the zeros of the function

$$u=x^{1/2}J_\alpha(x), \tag{1.4}$$

solution of the differential equation

$$\frac{d^2u}{dx^2}+\left[1+\frac{1-4\alpha^2}{4x^2}\right]u=0. \tag{1.5}$$

Another differential equation that we shall consider is the Airy equation

$$\frac{d^2v}{dx^2}-xv=0, \tag{1.6}$$

which is satisfied by the Airy function $\mathrm{Ai}(x)$ [1,11]. The negative zeros in decreasing order will be denoted by a_k, $k=1,2,\ldots$.

Interesting and general results can be easily obtained by applying the well-known Sturm comparison theorem [16, p. 19]. The first is given by the following theorem [16, (6.31.1)].

Theorem 1. *Let* $\alpha>-1$, *then*

$$\lambda_{n,k}^{(\alpha)}>\frac{j_{\alpha,k}^2}{v},\quad k=1,2,\ldots,n. \tag{1.7}$$

The constant $j_{\alpha,k}^2$ is the best possible in the sense that for a fixed k and for n arbitrary it cannot be replaced by a smaller one since

$$\lim_{n\to\infty} v\lambda_{n,k}^{(\alpha)}=j_{\alpha,k}^2.$$

An upper bound of $\lambda_{n,k}^{(\alpha)}$ of similar kind can also be obtained with the same Sturm method. More precisely, let ω be a positive constant such that $\omega<v$, then

$$\lambda_{n,k}^{(\alpha)}<\frac{j_{\alpha,k}^2}{v-\omega}, \tag{1.8}$$

if the expression on the right-hand side is not greater than ω. For a fixed k this is the case when n is sufficiently large. If we set

$$\omega=\frac{v}{2}-\sqrt{\left(\frac{v}{2}\right)^2-j_{\alpha,k}^2},$$

then, provided that $j_{\alpha,k} < \nu/2$, the inequality (1.8) yields

$$\lambda_{n,k}^{(\alpha)} < \frac{j_{\alpha,k}^2}{\nu/2 + \sqrt{(\nu/2)^2 - j_{\alpha,k}^2}}. \tag{1.9}$$

This bound, given by Riekstynsh [14, (6.104)] in a slightly different form, is a particular case of a more general formula for the zeros of confluent hypergeometric functions [21, p. 140].

An important result, which gives good results even for the zeros in the middle of oscillatory interval, is furnished by the following theorem (see [16, Theorem 6.31.3] and [1, p. 787]).

Theorem 2. *Let* $\alpha > -1$, *then*

$$\frac{(j_{\alpha,k}/2)^2}{n+(\alpha+1)/2} < \lambda_{n,k}^{(\alpha)} < \frac{2k+\alpha+1}{2n+\alpha+1}[2k+\alpha+1+((2k+\alpha+1)^2+1/4-\alpha^2)^{1/2}], \tag{1.10}$$

for $k = 1,2,\ldots,n$ *and* $n = 1,2,\ldots$. *In particular, for the first zero* $\lambda_{n,1}^{(\alpha)}$ *we have*

$$\frac{(j_{\alpha,1}/2)^2}{n+(\alpha+1)/2} < \lambda_{n,1}^{(\alpha)} \leqslant \frac{(\alpha+1)(\alpha+3)}{2n+\alpha+1}, \quad n = 1,2,\ldots . \tag{1.11}$$

The largest zero $\lambda_{n,n}^{(\alpha)}$ satisfies the inequality [16, (6.31.7)]

$$\lambda_{n,n}^{(\alpha)} < 2n+\alpha+1+[(2n+\alpha+1)^2+1/4-\alpha]^{1/2} \approx 4n, \tag{1.12}$$

which is weaker than the bound

$$\lambda_{n,n}^{(\alpha)} < 2n+\alpha-2+[1+4(n-1)(n+\alpha-1)]^{1/2}, \tag{1.13}$$

recently obtained by Ismail and Li [8] together with the lower bound for the first zero

$$\lambda_{n,1}^{(\alpha)} > 2n+\alpha-2-[1+4(n-1)(n+\alpha-1)]^{1/2}. \tag{1.14}$$

Notice that this last bound is more precise than the bound (1.7), which uses the first positive zero of $J_\alpha(x)$, only when the parameter α is large with respect to n.

Another more general result, for the kth zero $\lambda_{n,k}^{(\alpha)}$ of $L_n^{(\alpha)}(x)$ in terms of the zeros of Airy function $\mathrm{Ai}(x)$, is given by the following *Zernike–Hahn inequality* [16, (2.32.2)]

$$\lambda_{n,k}^{(\alpha)} < [(4n+2\alpha+2)^{1/2} + 2^{-1/3}(4n+2\alpha+2)^{-1/6}a_{n-k+1}]^2, \tag{1.15}$$

where $|\alpha| \geqslant 1/4$, $\alpha > -1$ and $k = 1,2,\ldots,n$.

Remark. Since the Hermite polynomials $H_n(x)$ can be entirely reduced to Laguerre polynomials with parameters $\alpha = \pm 1/2$ by means of the well-known formulas

$$\begin{aligned} H_{2m}(x) &= (-1)^m 2^{2m} m! L_m^{-1/2}(x^2), \\ H_{2m+1}(x) &= (-1)^m 2^{2m+1} m! L_m^{1/2}(x^2), \end{aligned} \tag{1.16}$$

their zeros are not explicitely considered in this paper.

2. Tricomi asymptotic approximations

We recall now three asymptotic approximations which give particularly good results.

Theorem 3 (Tricomi [18]). *Let $\alpha > -1$ and fixed, then as $n \to \infty$,*

$$\lambda_{n,k}^{(\alpha)} = \frac{j_{\alpha,k}^2}{v}\left[1 + \frac{j_{\alpha,k}^2 + 2(\alpha^2-1)}{3v^2}\right] + \mathrm{O}(n^{-5}), \tag{2.1}$$

for all the zeros belonging to the interval $0 < x \leqslant c/n$ with c fixed.

The next result involves only elementary functions and has been obtained by using the first two terms of a complete asymptotic expansion of $L_n^{(\alpha)}(x)$, in terms of ultraspherical polynomials, established by Tricomi [20].

Theorem 4 (Tricomi [20]). *Let $T_{n,k}$ be the root of the equation*

$$x - \sin x = \frac{4n-4k+3}{v}\pi, \quad 0 < x < \pi. \tag{2.2}$$

Set

$$t_{n,k} = \cos^2\frac{T_{n,k}}{2}, \tag{2.3}$$

then we have, as $n \to \infty$,

$$\lambda_{n,k}^{(\alpha)} = vt_{n,k} - \frac{1}{3v}\left[\frac{5}{4(1-t_{n,k})^2} - \frac{1}{1-t_{n,k}} - 1 + 3\alpha^2\right] + \mathrm{O}(n^{-3}), \tag{2.4}$$

for all the zeros $\lambda_{n,k}^{(\alpha)}$ belonging to the interval $[av, bv]$ with a and b fixed $(0 < a < b < 1)$, or, in other words, for $=[pn], [pn]+1, \ldots, [qn]$, p and q being $(0 < p < q < 1)$ two fixed numbers.

We notice that (2.4) was obtained by Tricomi with a different bound for the error term; in Section 4 we shall prove that the correct bound is the one given here. Furthermore, we remark that the above approximation could be improved by considering more terms of Tricomi expansion, but the formula becomes uselessly very complicated. Indeed, formula (2.4) gives good results in the interval of validity but it furnishes also good starting values in iterative procedures for the numerical evaluation of the zeros which lie outside such interval.

Finally, we recall an approximation of the large zeros.

Theorem 5 (Tricomi [19]). *Let $\alpha > -1$ and let a_m be the mth zero of the Airy function* $\mathrm{Ai}(x)$. *Then*

$$\lambda_{n,n-m+1}^{(\alpha)} = v + 2^{2/3}a_m v^{1/3} + \tfrac{1}{5}2^{4/3}a_m^2 v^{-1/3} + \mathrm{O}(n^{-1}), \tag{2.5}$$

as $n \to \infty$ and m is fixed.

This last asymptotic result will be improved in Section 4.

3. Uniform approximations for fixed values of α

In this Section we shall recall some results obtained by using two complete uniform asymptotic expansions for $v \to \infty$ and fixed $\alpha > -1$, due to Frenzen and Wong [4,22, Chapter VII], of the polynomial $L_n^{(\alpha)}(x)$. Such expansion involves the Bessel functions $J_\alpha(x)$ and $J_{\alpha+1}(x)$, and the Airy function $\mathrm{Ai}(x)$ and its derivative respectively. Furthermore, they hold uniformly in the two overlapping intervals $0 \leqslant x \leqslant bv$ and $av \leqslant x \leqslant v$, with $0 < a < b < 1$ fixed as $n \to \infty$, that is, they cover all the interval containing the zeros of $L_n^{(\alpha)}(x)$.

Theorem 6 (Gatteschi [6]). *Let $-1 < \alpha < 1$. Let $U_{n,k}$ be the root of the equation*

$$x - \sin x = \pi - \frac{4j_{\alpha,k}}{v}, \quad 0 < x < \pi, \tag{3.1}$$

then

$$\lambda_{n,k}^{(\alpha)} < v\cos^2\frac{U_{n,k}}{2}, \quad k = 1,2,\ldots,n. \tag{3.2}$$

Theorem 7 (Gatteschi [6]). *Let $V_{n,k}$ be the root of the equation*

$$x - \sin x = \frac{8}{3v}(-a_{n-k+1})^{3/2}, \quad 0 < x < \pi, \tag{3.3}$$

where a_j is the jth zero of the Airy function $\mathrm{Ai}(x)$. *Then we have*

$$\lambda_{n,k}^{(\alpha)} > v\cos^2\frac{V_{n,k}}{2}, \quad if \ -\frac{1}{2} \leqslant \alpha \leqslant \frac{1}{2}, \tag{3.4}$$

$$\lambda_{n,k}^{(\alpha)} < v\cos^2\frac{V_{n,k}}{2}, \quad if \ -1 < \alpha \leqslant -\frac{2}{3} \quad or \quad \alpha \geqslant \frac{2}{3}, \tag{3.5}$$

for $k = 1.2,\ldots,n$.

We remark that inequality (3.5) furnishes a very sharp upper bound for the interval containing all the zeros of $L_n^{(\alpha)}(x)$. E.g., for the largest zero of $\alpha = 1$ we find $\lambda_{n,n}^{(1)} < 31.8984\ldots$, while the exact value is $\lambda_{n,n}^{(1)} = 31.6828\ldots$ and the bounds given by (1.12), (1.13) and (1.15) are $\lambda_{n,n}^{(1)} < 43.9829\ldots$, $\lambda_{n,n}^{(1)} < 38$ and $\lambda_{n,n}^{(1)} < 31.8725\ldots$, respectively.

Theorem 8 (Gatteschi [7]). *Let $\alpha > -1$ and let $U_{n,k}$ have the same meaning as in Theorem 6. Set*

$$u_{n,k} = \cos^2\frac{U_{n,k}}{2}, \tag{3.6}$$

then we have, as $n \to \infty$,

$$\lambda_{n,k}^{(\alpha)} = vu_{n,k} - \frac{1}{2v}\left[\frac{(1-4\alpha^2)v}{2j_{\alpha,k}}\left(\frac{u_{n,k}}{1-u_{n,k}}\right)^{1/2} + \frac{4\alpha^2-1}{2} + \frac{u_{n,k}}{1-u_{n,k}} + \frac{5}{6}\left(\frac{u_{n,k}}{1-u_{n,k}}\right)^2\right] + \mathrm{O}(n^{-3}). \tag{3.7}$$

The O*-term is uniformly bounded for all the values of* $k = 1, 2, \ldots, [qn]$ *where* q *is a fixed number in the interval* $(0, 1)$.

Theorem 9 (Gatteschi [7]). *Let* $\alpha > -1$ *and let* $V_{n,k}$ *have the same meaning as in Theorem* 7. *Set*

$$v_{n,k} = \cos^2 \frac{V_{n,k}}{2}, \tag{3.8}$$

then we have, as $n \to \infty$,

$$\lambda_{n,k}^{(\alpha)} = \nu v_{n,k} + \frac{1}{\nu}\left[\frac{5\nu}{24}(-a_{n-k+1})^{-3/2}\left(\frac{v_{n,k}}{1 - v_{n,k}}\right)^{1/2} + \frac{1}{4} - \alpha^2 - \frac{1}{2}\frac{v_{n,k}}{1 - v_{n,k}} - \frac{5}{12}\left(\frac{v_{n,k}}{1 - v_{n,k}}\right)^2\right] + \mathrm{O}(n^{-3}). \tag{3.9}$$

The O*-term is uniformly bounded for all the values of* $k = [pn], [pn] + 1, \ldots, n$ *where* p *is a fixed number in the interval* $0, 1$.

In order to compare the approximations $\bar{\lambda}_{n,k}^{(\alpha)}$ of the zeros $\lambda_{n,k}^{(\alpha)}$ obtained by using (3.7), (3.9) and Tricomi approximation (2.4) it is convenient to define the accuracy $\rho(n, \alpha, k)$, that is the number of correct decimal digits in the approximation,

$$\rho(n, \alpha, k) = -\log_{10}\left|\frac{\lambda_{n,k}^{(\alpha)} - \bar{\lambda}_{n,k}^{(\alpha)}}{\lambda_{n,k}^{(\alpha)}}\right|. \tag{3.10}$$

The accuracies $\rho_B(\alpha)$, $\rho_A(\alpha)$ and $\rho_T(\alpha)$, associated to the approximations obtained by omitting the O-terms in (3.7), (3.9) and (2.4), respectively, are given for $n = 16$ and $\alpha = 0, 1/2$ and 1 in Table 1.

4. Remarks and other results

The approximation (2.4) given by Tricomi can be derived from the uniform formula (3.7) and it can be shown that the bound for the error term is $\mathrm{O}(n^{-3})$.

To do this, we note that the numbers $t_{n,k}$ and $u_{n,k}$ defined by (2.4) and (3.6) are the roots in the interval $0 < t < 1$ of the two equations

$$\arccos\sqrt{t} - \sqrt{t - t^2} = \frac{4n - 4k + 3}{2\nu}\pi,$$

$$\arccos\sqrt{t} - \sqrt{t - t^2} = \frac{\pi}{2} - \frac{2j_{\alpha,k}}{\nu}, \tag{4.1}$$

respectively. Further, recalling that [1, p. 371], as $k \to \infty$,

$$j_{\alpha,k} = \left(k + \frac{\alpha}{2} - \frac{1}{4}\right)\pi - \frac{4\alpha^2 - 1}{8(k + \alpha/2 - \frac{1}{4})\pi} + \mathrm{O}(k^{-3}) \tag{4.2}$$

Table 1
Accuracies $\rho_B(\alpha)$, $\rho_A(\alpha)$ and $\rho_T(\alpha)$ for $\alpha = 0, 1/2$ and 1

k	$\rho_B(0)$	$\rho_A(0)$	$\rho_T(0)$	$\rho_B(1/2)$	$\rho_A(1/2)$	$\rho_T(1/2)$	$\rho_B(1)$	$\rho_A(1)$	$\rho_T(1)$
1	6.96	2.49	2.49	7.16	7.75	7.16	9.21	3.13	3.13
2	6.95	3.84	3.84	7.15	7.40	7.15	8.67	4.13	4.13
3	6.93	4.59	4.59	7.12	7.54	7.12	8.33	4.76	4.76
4	6.90	5.11	5.12	7.08	7.68	7.08	8.06	5.21	5.21
5	6.86	5.51	5.52	7.03	7.81	7.03	7.83	5.57	5.55
6	6.80	5.82	5.86	6.96	7.93	6.96	7.62	5.86	5.83
7	6.74	6.09	6.17	6.88	8.04	6.88	7.42	6.11	6.04
8	6.65	6.31	6.51	6.78	8.14	6.78	7.23	6.32	6.19
9	6.55	6.51	7.08	6.66	8.24	6.66	7.02	6.50	6.28
10	6.42	6.68	6.98	6.52	8.33	6.52	6.80	6.67	6.29
11	6.26	6.83	6.29	6.34	8.41	6.34	6.56	6.82	6.22
12	6.05	6.97	6.17	6.11	8.50	6.11	6.28	6.96	6.06
13	5.77	7.10	5.84	5.82	8.58	5.82	5.95	7.09	5.81
14	5.39	7.23	5.43	5.43	8.66	5.43	5.52	7.21	5.43
15	4.82	7.35	4.84	4.85	8.74	4.85	4.90	7.33	4.85
16	3.76	7.47	3.77	3.78	8.84	3.78	3.81	7.46	3.78

the second equation in (4.1) becomes

$$\arccos\sqrt{t} - \sqrt{t-t^2} = \frac{4n-4k+3}{2v}\pi + \frac{4\alpha^2-1}{4(k+\alpha/2-\frac{1}{4})v\pi} + \frac{1}{v}\mathrm{O}(k^{-3}).$$

It follows that, if we assume as in Theorem 4 $k = [pn], [pn]+1, \ldots, [qn]$, where p and q are two fixed numbers in $(0,1)$, then

$$\arccos\sqrt{t} - \sqrt{t-t^2} = \frac{4n-4k+3}{2v}\pi + \frac{4\alpha^2-1}{4(k+\alpha/2-\frac{1}{4})v\pi} + \mathrm{O}(n^{-4}). \tag{4.3}$$

So we have

$$u_{n,k} = t_{n,k} + \varepsilon_{n,k}, \quad \varepsilon_{n,k} = \mathrm{O}(n^{-2}), \tag{4.4}$$

as $n \to \infty$ and with the above specified values of k. More precisely, since $t_{n,k}$ satisfies the first of (4.1), a Taylor expansion of the right-hand side of (4.3) gives

$$-\varepsilon_{n,k}\sqrt{\frac{1-t_{n,k}}{t_{n,k}}} = \frac{4\alpha^2-1}{4(k+\alpha/2-\frac{1}{4})v\pi} + \mathrm{O}(n^{-4}),$$

that is, from (4.4),

$$u_{n,k} = t_{n,k} - \sqrt{\frac{t_{n,k}}{1-t_{n,k}}}\frac{4\alpha^2-1}{4(k+\alpha/2-\frac{1}{4})v\pi} + \mathrm{O}(n^{-4}). \tag{4.5}$$

We now observe that the relationships (4.2) and (4.4) allow us to write (3.7) in the form

$$\lambda_{n,k}^{(\alpha)} = \nu u_{n,k} - \frac{1}{2\nu}\left[\frac{2(1-4\alpha^2)\nu}{(4k+2\alpha-1)\pi}\left(\frac{t_{n,k}}{1-t_{n,k}}\right)^{1/2} + \frac{4\alpha^2-1}{2} + \frac{t_{n,k}}{1-t_{n,k}} + \frac{5}{6}\left(\frac{t_{n,k}}{1-t_{n,k}}\right)^2\right] + \mathrm{O}(n^{-3}).$$

Next, substituting $u_{n,k}$ with its value given by (4.5), we have

$$\lambda_{n,k}^{(\alpha)} = \nu t_{n,k} - \frac{1}{2\nu}\left[\frac{4\alpha^2-1}{2} + \frac{t_{n,k}}{1-t_{n,k}} + \frac{5}{6}\left(\frac{t_{n,k}}{1-t_{n,k}}\right)^2\right] + \mathrm{O}(n^{-3}) \tag{4.6}$$

that coincides with Tricomi formula (2.4), as it is easily verified.

Similarly, we prove the asymptotic equivalence between the right-hand sides of (2.4) and (3.9) when $k=[pn],[pn]+1,\ldots,[qn]$, where p as $n\to\infty$. The proof is based on the asymptotic formula for the large zeros of the Airy function $\mathrm{Ai}(x)$ [1, p. 450]

$$a_s = -z^{2/3}\left[1+\frac{5}{48}\frac{1}{z^2}+\mathrm{O}(z^{-4})\right] = -\left[z+\frac{5}{32}\frac{1}{z}+\mathrm{O}(z^{-3})\right]^{2/3},$$

where $z=3\pi(4s-1)/8$ and $s\to\infty$.

We also notice that the other Tricomi formulas (2.1) and (2.5) can be derived as particular cases of (3.7) and (3.9). In what follows we show that (3.9) yields, as particular case, an interesting improvement of (2.5). We start observing that the numbers $v_{n,k}$, $k=1,2,\ldots,n$, defined by (3.8) satisfy the equation,

$$\left(\tfrac{3}{4}\arccos\sqrt{t} - \tfrac{3}{4}\sqrt{t-t^2}\right)^{2/3} = -\frac{a_{n-k+1}}{\nu^{2/3}}.$$

Should we be interested in the large zeros of Laguerre polynomials, we assume t in the interval $1-c/\nu \leqslant t \leqslant 1$ where c is a fixed positive constant as $\nu\to\infty$. It follows, by putting

$$y_k = -\frac{a_{n-k+1}}{\nu^{2/3}}, \tag{4.7}$$

that $v_{n,k}$ is obtained by inverting the series

$$\begin{aligned} y_k = {} & \frac{1}{2}2^{1/3}(1-t) + \frac{1}{10}2^{1/3}(1-t)^2 + \frac{17}{350}2^{1/3}(1-t)^3 + \frac{473}{15750}2^{1/3}(1-t)^4 \\ & + \frac{63383}{3031875}2^{1/3}(1-t)^5 + \frac{3079471}{197071875}2^{1/3}(1-t)^6 + \frac{760436123}{62077640625}2^{1/3}(1-t)^7 \\ & + \frac{7490269453}{753799921875}2^{1/3}(1-t)^8 + \mathrm{O}((1-t)^9). \end{aligned}$$

Hence, as $y \to 0$,

$$v_{n,k} = 1 - \left(2^{2/3}y - \frac{2}{5}2^{1/3}y_k^2 - \frac{12}{175}y_k^3 - \frac{92}{7875}2^{2/3}y_k^4 - \frac{15152}{3031875}2^{1/3}y_k^5 - \frac{52688}{21896875}y_k^6 \right.$$
$$\left. + \frac{38689472}{62077540625}2^{2/3}y_k^7 - \frac{358175648}{1055319890625}2^{1/3}y_k^8 + \mathrm{O}(y_k^9)\right). \tag{4.8}$$

Substituting in the right-hand side of (3.9), taking into account that y_k is given by (4.7), and with large use of symbolic calculation (Maple V), we obtain the following improvement of Tricomi asymptotic formula (2.5).

Theorem 10. *Let $\alpha > -1$ and let a_m be the mth zero of the Airy function* $\mathrm{Ai}(x)$. *Then*

$$\lambda_{n,n-k+1}^{(\alpha)} = v + 2^{2/3}a_k v^{1/3} + \frac{1}{5}2^{4/3}a_k^2 v^{-1/3} + \left(\frac{11}{35} - \alpha^2 - \frac{12}{175}a_k^3\right)v^{-1}$$
$$+ \left(\frac{16}{1575}a_k + \frac{92}{7875}a_k^4\right)2^{2/3}v^{-5/3} - \left(\frac{15152}{3031875}a_k^5 + \frac{1088}{121275}a_k^2\right)2^{1/3}v^{-7/3} + \mathrm{O}(v^{-3}), \tag{4.9}$$

as $n \to \infty$ and k is fixed.

By omitting the O-term in (4.9) we obtain very good approximations specially for the last few zeros of $L_n^{(\alpha)}(x)$. For instance, in the case $n = 16$ and $\alpha = 1/4$ we find for the last three zeros, in decreasing order, the approximations

52.1605848856, 42.3675443783, 34.9818546717,

while the correct values are

52.1605847023, 42.3675835594, 34.9821169778.

Note that the previous Triomi formula gives

52.1431243384, 42.2882400930, 34.7870980822.

The connection of the approximation formulas (2.1), (2.4) and (2.5), established by Tricomi, and the new formula (4.9) with the two uniform formulas (3.7) and (3.9) is visualized in Fig. 1, which refers to the accuracies ρ_{TB}, ρ_T, ρ_{TA}, ρ_{GA}, ρ_B and ρ_A, obtained in the case $n = 24$ and $\alpha = 0$.

Since the uniform approximations (3.7) and (3.9) furnish good numerical results, it could be interesting to find upper bounds for their error terms. This problem presents many difficulties even if we have more specified information on the error terms of the expansions considered to construct (3.7) and (3.9).

Pittaluga and Sacripante [13] have derived interesting inequalities from (3.7) in the particular cases $\alpha = -\frac{1}{2}$ and $\alpha = \frac{1}{2}$, that is for the zeros of Hermite polynomials.

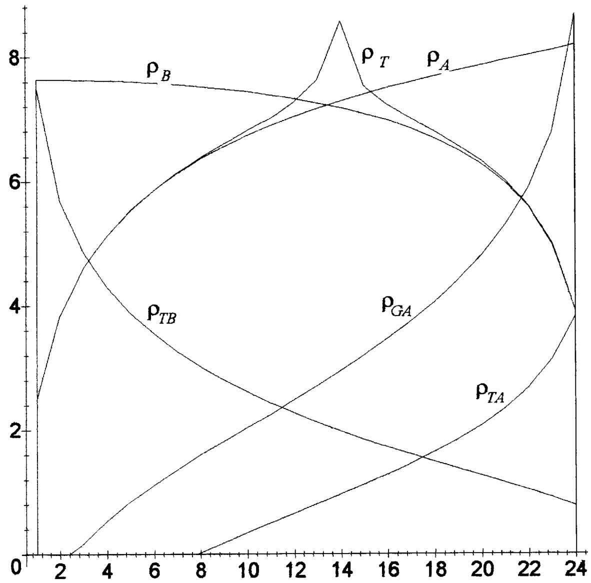

Fig. 1. The accuracies for $n = 24$ and $\alpha = 0$.

Theorem 11. *Let* $\lambda_{n,k}^{(\mp 1/2)}$, $k = 1,2,\ldots,n$, *be the zeros of the Laguerre polynomials* $L_n(x)^{(\mp 1/2)}$ *and* $\xi_{n,k}^{(\mp 1/2)}$, $k = 1,2,\ldots,n$, *the roots of the equation*

$$x - \sin x = \frac{4n - 4k + 3}{4n \mp 1 + 2}\pi, \quad 0 < x < \pi.$$

Set

$$\tau_{n,k}^{(\mp 1/2)} = \cos^2 \frac{\xi_{n,k}^{(\mp 1/2)}}{2},$$

then we have

$$\lambda_{n,k}^{(\mp 1/2)} > (4n \mp 1 + 2)\tau_{n,k}^{(\mp 1/2)} - \frac{1}{2(4n \mp 1 + 2)} \cdot \left[\frac{\tau_{n,k}^{(\mp 1/2)}}{1 - \tau_{n,k}^{(\mp 1/2)}} - \frac{5}{6}\left(\frac{\tau_{n,k}^{(\mp 1/2)}}{1 - \tau_{n,k}^{(\mp 1/2)}} \right)^2 \right], \qquad (4.10)$$

for $k = 1,2,\ldots,n$.

More generally, many numerical evaluations lead to the following conjectures.

Conjecture 12. *Let* $-1/2 \leqslant \alpha \leqslant 1/2$ *and* $u_{n,k}$ *have the same meaning as in* (3.7). *Then*

$$\lambda_{n,k}^{(\alpha)} > \nu u_{n,k} - \frac{1}{2\nu}\left[\frac{(1 - 4\alpha^2)\nu}{2j_{\alpha,k}} \left(\frac{u_{n,k}}{1 - u_{n,k}} \right)^{1/2} + \frac{4\alpha^2 - 1}{2} + \frac{u_{n,k}}{1 - u_{n,k}} + \frac{5}{6}\left(\frac{u_{n,k}}{1 - u_{n,k}} \right)^2 \right], \qquad (4.11)$$

for $k = 1,2,\ldots,n$.

Conjecture 13. *Let* $-1/2 \leqslant \alpha \leqslant 1/2$ *and* $v_{n,k}$ *have the same meaning as in* (3.9). *Then*

$$\lambda_{n,k}^{(\alpha)} < \nu v_{n,k} + \frac{1}{\nu}\left[\frac{5\nu}{24}(-a_{n-k+1})^{-3/2}\left(\frac{v_{n,k}}{1-v_{n,k}}\right)^{1/2} + \frac{1}{4} - \alpha^2 - \frac{1}{2}\frac{v_{n,k}}{1-v_{n,k}} - \frac{5}{12}\left(\frac{v_{n,k}}{1-v_{n,k}}\right)^2\right], \tag{4.12}$$

for $k = k_0, k_0+1, \ldots, n$, *where* $1 \leqslant k_0 < n$ *depends on* α *and* n.

In other words, the above upper bound (4.12) does not generally hold for all the n zeros of $L_n^{(\alpha)}(x)$, but a certain number of the first zeros must be eventually excluded. This number is small with respect to n, for instance in the worst case of $\alpha = -1/2$ occurs that:

- If $n = 1, 2, \ldots, 9$ then (4.12) holds for $k = 1, 2, \ldots, n$,
- If $n = 10, 11, \ldots, 27$ then (4.12) holds for $k = 2, 3, \ldots, n$,
- If $n = 28, 29, \ldots, 47$ then (4.12) holds for $k = 3, 4, \ldots, n$,
- If $n = 48, 49, \ldots, 65$ then (4.12) holds for $k = 4, 5, \ldots, n$.

Remark. Probably, the last conjecture holds with $k_0 = 1$ if $-0.4999 \leqslant \alpha \leqslant 1/2$.

5. Large values of the parameter α

Important results on the approximation of the zeros $\lambda_{n,k}^{(\alpha)}$ when the parameter α is large with respect to n, or when both parameters are large, may be obtained by using asymptotic representations of $L_n^{(\alpha)}(x)$ in terms of Hermite polynomials $H_n(x)$. The first of these representations is given by the following limit [10, p. 25]

$$\lim_{\alpha\to\infty}\left[\mathrm{e}^{-n/2}L_n^{(\alpha)}(\alpha + t\sqrt{\alpha})\right] = 2^{-n/2}H_n\left(\frac{t}{\sqrt{2}}\right), \tag{5.1}$$

which can be easily justified observing that, by putting $x = \alpha + t\sqrt{\alpha}$, the Laguerre equation

$$xy'' + (\alpha + 1 - x)y' + ny = 0, \quad y = L_n^{(\alpha)}(x),$$

becomes

$$\left(1 + \frac{t}{\sqrt{\alpha}}\right)\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \left(\frac{1}{\sqrt{\alpha}} - t\right)\frac{\mathrm{d}y}{\mathrm{d}t} + ny = 0, \quad y = L_n^{(\alpha)}(\alpha + t\sqrt{\alpha}),$$

that, for large values of α, is close to the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - t\frac{\mathrm{d}y}{\mathrm{d}t} + ny = 0,$$

satisfied by $y = H_n(t/\sqrt{2})$. It follows that, denoting with $h_{n,k}$ the kth zero of $H_n(x)$, we have

$$\lambda_{n,k}^{(\alpha)} \approx \alpha + h_{n,k}\sqrt{2\alpha}, \quad \alpha \to \infty, \quad k = 1, 2, \ldots, n. \tag{5.2}$$

This very simple approximation formula has been improved by Calogero [2,9] which has shown that

$$\lambda_{n,k}^{(\alpha)} = \alpha + \sqrt{2\alpha} + \tfrac{1}{3}(1 + 2n + 2h_{n,k}^2) + \mathrm{O}(\alpha^{-1/2}) \quad \alpha \to \infty. \tag{5.3}$$

5.1. Temme asymptotic approximation

The formulas (5.2) and (5.3) give numerically meaningful results only if $\alpha \gg n$. Better results can be obtained by using an asymptotic representation given by Temme [17]. Such representation is based on Olver's [12] asymptotic approximations of Whittaker functions in terms of parabolic cylinder functions which, in the case of Laguerre polynomials, reduce to Hermite polynomials. An important role is played by the Liouville–Green transformation of differential equations [11, Chapter 6].

In this section, we write the differential equation (1.2) in the form

$$\frac{d^2 y}{dt^2} + \left[\frac{1}{4t^2} - \frac{v^2}{4t^2}(t-t_1)(t-t_2)\right] y = 0, \tag{5.4}$$

where

$$t_1 = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-\tau^2}, \quad t_2 = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-\tau^2}, \tag{5.5}$$

being

$$\tau = \frac{2\alpha}{v}. \tag{5.6}$$

With the new independent variable $\eta = \eta(t)$ and dependent variable Y defined by

$$(\mu^2-\eta^2)^{1/2}\frac{d\eta}{dt} = \frac{2R(t)}{t^2}, \quad Y = \left(\frac{d\eta}{dt}\right)^{1/2} y, \tag{5.7}$$

where

$$R(t) = \sqrt{(t-t_1)(t_2-t)} = \frac{1}{2}\sqrt{4t-4t^2-\tau^2} \tag{5.8}$$

and

$$\mu = \sqrt{2(1-\tau)}, \tag{5.9}$$

the Liouville–Green transformation leads [17] to the construction of the required solution of (5.4) in terms of Hermite polynomials.

More precisely, Temme [17] has shown that in the oscillatory interval $0 < t < 1$, that is in the interval corresponding to the one containing the zeros $\lambda_{n,k}^{(\alpha)} = vt_{n,k}$ of $L_n^{(\alpha)}(x)$, for $v \to \infty$,

$$\begin{aligned} L_n^{(\alpha)}(vt) \sim{}& \frac{(-1)^n}{n!} 2^{-n/2-1/4} v^{-\alpha/2-1/4} t^{-\alpha/2} \exp\left(\frac{v}{2}t - \frac{v}{8}\eta^2 - \frac{n}{2} - \alpha - \frac{1}{2}\right) \\ &\times \left(n+\alpha+\frac{1}{2}\right)^{n+\alpha+1/2} \left[\frac{\mu^2-\eta^2}{(t-t_1)(t_2-t)}\right]^{1/4} H_n\left(\frac{\eta}{2}\sqrt{v}\right), \end{aligned} \tag{5.10}$$

where η is the root in the interval $-\mu \leqslant \eta \leqslant \mu$ of the equation

$$\frac{\eta}{2}\sqrt{\mu^2-\eta^2} + \frac{\mu^2}{2}\arcsin\frac{\eta}{\mu} = 2R - \tau\arctan\frac{2t-\tau^2}{2\tau R} - \arctan\frac{1-2t}{2R}. \tag{5.11}$$

This representation yields the following statement.

Table 2
Temme approximation

k	$\alpha=1$	$\alpha=10$	$\alpha=100$	$\alpha=1000$	k	$\alpha=1$	$\alpha=10$	$\alpha=100$	$\alpha=1000$
1	2.30	3.66	5.57	7.62	6	3.52	4.16	5.73	7.66
2	2.72	3.79	5.61	7.63	7	3.65	4.24	5.75	7.67
3	2.99	3.90	5.64	7.64	8	3.77	4.31	5.78	7.68
4	3.20	3.99	5.67	7.65	9	3.88	4.39	5.81	7.69
5	3.37	4.08	5.70	7.66	10	4.00	4.47	5.85	7.70

Theorem 14. *Let $h_{n,k}$, $k=1,2,\ldots,n$ be the zeros of the Hermite polynomial $H_n(x)$ and set*

$$\eta_{n,k}=\frac{2h_{n,k}}{\sqrt{v}},\quad k=1,2,\ldots,n. \tag{5.12}$$

Substituting in (5.11) *and inverting with respect to t we obtain $t=t_{n,k}$, $k=1,2,\ldots,n$. It follows that, for the zero $\lambda_{n,k}^{(\alpha)}$ of $L_n^{(\alpha)}(x)$, we have*

$$\lambda_{n,k}^{(\alpha)}\sim vt_{n,k},\quad n\to\infty. \tag{5.13}$$

This estimate holds uniformly with respect to $k=1,2,\ldots,n$ and $\alpha\geqslant 0$.

The Table 2 shows the accuracies obtained applying (5.13). The numerical results suggest to conjecture that Temme approximation is an upper bound.

5.2. The Smith approximation

An approximate representation of the Laguerre polynomial $L_n^{(\alpha)}(x)$ for large values of n and α has been given recently by Smith [15]. More precisely, referring to the function $L_n^{(na)}(nt)$ and putting

$$a>0,\quad \Delta=\sqrt{4t-(a-t)^2}, \tag{5.14}$$

$$\theta_0=\arccos\frac{a-t}{2\sqrt{t}},\quad \theta_1=\arctan\frac{\Delta}{a+t} \tag{5.15}$$

the steepest-descent method leads to the asymptotic estimate, for $n\to\infty$,

$$L_n^{(na)}(nt)\sim g(n,t)\sin\left[\left(n+\frac{1}{2}\right)\theta_0+n\frac{\Delta}{2}-\left(n(1+a)+\frac{1}{2}\right)\theta_1+\frac{\pi}{4}\right], \tag{5.16}$$

where

$$g(n,t)=\sqrt{\frac{2\sqrt{1+a}}{\pi n\Delta}}\exp\left\{-\frac{n}{2}[a-t-(1+\alpha)\log(1+a)+a\log t]\right\},$$

valid in the region where Δ is real and not too small.

So, we obtain the following algorithm for the zeros $\lambda_{n,k}^{(\alpha)}$.

Table 3
Smith approximation

k	$\alpha = 1$	$\alpha = 10$	$\alpha = 100$	$\alpha = 1000$	k	$\alpha = 1$	$\alpha = 10$	$\alpha = 100$	$\alpha = 1000$
1	1.71	2.27	2.92	3.51	6	3.05	3.38	4.21	5.40
2	2.26	2.75	3.42	4.05	7	3.15	3.47	4.46	5.26
3	2.57	3.00	3.70	4.38	8	3.25	3.60	5.19	4.67
4	2.78	3.17	3.89	4.64	9	3.45	4.10	3.98	4.22
5	2.93	3.28	4.05	4.91	10	3.29	3.15	3.24	3.61

Theorem 15. *Let $t_{n,k}$, $k = 1, 2, \ldots, n$ be the roots of the equations*

$$\left(n + \frac{1}{2}\right)\theta_0 + n\frac{\Delta}{2} - \left[n(1 + a) + \frac{1}{2}\right]\theta_1 + \frac{\pi}{4} = k\pi, \quad k = 1, 2, \ldots, n. \tag{5.17}$$

Then we have

$$\lambda_{n,k}^{(\alpha)} \sim nt_{n,k}, \quad n \to \infty. \tag{5.18}$$

The accuracies related to this last approximation formula are shown in Table 3. The comparison with the Table 2 shows that Smith results are not so good as the ones obtained by using Temme procedure. But Smith formula is much simpler and involves only elementary functions.

6. Two new asymptotic approximations

In this section, we consider, without proof, two asymptotic approximations of the zeros $\lambda_{n,k}^{(\alpha)}$, in terms of the zeros $j_{\alpha,k}$ and a_k, respectively. Such representations have been recently obtained, in a joint paper with Gabutti [5], in the more general case of the zeros of Whittaker's confluent hypergeometric functions by using the complete expansions, in terms of Bessel and Airy functions, established by Dunster [3].

6.1. Bessel-type approximations

With the new independent variable $\zeta = \zeta(t)$ and dependent variable Y defined by

$$\frac{d\zeta}{dt} = \frac{\zeta}{t}\frac{4R(t)}{(\zeta - 4\tau^2)^{1/2}}, \quad Y = \left(\frac{d\zeta}{dt}\right)^{1/2} y, \tag{6.1}$$

where the same notations (5.5), (5.6) and (5.8) are used, the Liouville–Green transformation brings Eq. (5.4) in an equation which is close to the Bessel equation. It follows that we can now apply the method developed by Olver [11] for constructing asymptotic approximations, in terms of Bessel functions. For our purposes it is sufficient to refer to the case $t_1 \leqslant t < t_2$ and correspondingly $\zeta \geqslant 4\tau^2$.

Table 4
First Bessel-type approximation

k	$\alpha=1$	$\alpha=10$	$\alpha=100$	$\alpha=1000$	k	$\alpha=1$	$\alpha=10$	$\alpha=100$	$\alpha=1000$
1	3.46	3.67	4.18	4.72	6	3.29	3.45	3.85	4.34
2	3.44	3.65	4.13	4.66	7	3.22	3.36	3.74	4.22
3	3.42	3.61	4.07	4.59	8	3.10	3.23	3.59	4.06
4	3.39	3.57	4.01	4.52	9	2.91	3.03	3.37	3.83
5	3.35	3.52	3.94	4.43	10	2.53	2.64	2.95	3.40

Solving the first differential equation in (6.1) we obtain [3,5]

$$\sqrt{\zeta-4\tau^2}-2\tau\arctan\frac{\sqrt{\zeta-4\tau^2}}{2\tau}=2R(t)-\tau\arctan\frac{2t-\tau^2}{2\tau R(t)}-\arctan\frac{1-2t}{2R(t)}+\frac{1}{2}\pi(1-\tau). \quad (6.2)$$

Then we can derive, from a two-term representation [5] of the Whittaker function $M_{\kappa,\mu}(z)$ in the particular case $\kappa=n+(\alpha+1)/2$, $\mu=\alpha/2$, the following asymptotic approximation for the zeros $\lambda_{n,k}^{(\alpha)}$ of $L_n^{(\alpha)}(x)$.

Theorem 16. *Let $j_{\alpha,k}$, $k=1,2,\ldots$, be the zeros of the Bessel function $J_\alpha(x)$ and set*

$$\zeta_{n,k}=\left(\frac{4j_{\alpha,k}}{v}\right)^2,\quad k=1,2,\ldots,n. \quad (6.3)$$

Substituting $\zeta=\zeta_{n,k}$ in (6.2) *and inverting with respect to t we obtain $t_{n,k}$, which yields a first approximation for the zeros $\lambda_{n,k}^{(\alpha)}$ of $L_n^{(\alpha)}(x)$. More precisely we have, as $v\to\infty$,*

$$\lambda_{n,k}^{(\alpha)}=vt_{n,k}+O(v^{-1}),\quad k=1,2,\ldots\ . \quad (6.4)$$

Furthermore, an improved formula is given by

$$\lambda_{n,k}^{(\alpha)}=vt_{n,k}-\frac{4t_{n,k}}{vR(t_{n,k})}\left[\frac{1}{4}\frac{1}{(\zeta_{n,k}-4\tau^2)^{1/2}}+\frac{5}{3}\frac{\tau^2}{(\zeta_{n,k}-4\tau^2)^{3/2}}\right.$$
$$\left.+\frac{1}{96}\frac{8t_{n,k}^3+12(\tau^2-2)t_{n,k}^2+6t_{n,k}-\tau^2(2\tau^2-1)}{(\tau^2-1)R^3(t_{n,k})}\right]+O(v^{-3}). \quad (6.5)$$

The bounds for the error terms in (6.4) *and* (6.5) *hold uniformly for all the values of $k=1,2,\ldots,[qn]$ where q is a fixed positive number in* (0.1).

Tables 4 and 5 refer respectively to the accuracies associated to the approximations obtained by evaluating the zeros $\lambda_{10,k}^{(\alpha)}$, $k=1,2,\ldots,10$, of $L_{10}^{(\alpha)}(x)$ for $\alpha=1,10,100$ and 1000 using (6.4) and (6.5) without the error terms.

The many evaluations we have done suggest to conjecture that the first and the second approximations are respectively upper and lower bounds for all the zeros of $L_n^{(\alpha)}(x)$. Unfortunately, the conjecture appears very difficult to be proved.

Table 5
Second Bessel-type approximation

k	$\alpha=1$	$\alpha=10$	$\alpha=100$	$\alpha=1000$	k	$\alpha=1$	$\alpha=10$	$\alpha=100$	$\alpha=1000$
1	6.24	6.59	7.19	7.74	6	5.77	5.96	6.39	6.88
2	6.20	6.52	7.07	7.60	7	5.53	5.70	6.10	5.58
3	6.14	6.43	6.94	7.46	8	5.19	5.39	5.70	6.17
4	6.06	6.31	6.79	7.30	9	4.65	4.72	5.12	5.58
5	5.94	6.16	6.61	7.11	10	3.62	3.73	4.04	4.49

6.2. Airy-type approximations

We take a new independent variable, which for convenience we shall denote by $\hat{\zeta}=\hat{\zeta}(t)$, and a dependent variable $\hat{Y}$, defined by

$$\frac{\mathrm{d}\hat{\zeta}}{\mathrm{d}t}=\frac{2}{t}\,\frac{R(t)}{\hat{\zeta}^{1/2}},\quad \hat{Y}=\left(\frac{\mathrm{d}\hat{\zeta}}{\mathrm{d}t}\right)^{1/2}y. \tag{6.6}$$

Then, the effect of Liouville–Green transformation is to carry (5.4) in an equation whose solutions can be approximated in terms of Airy functions.

In the interval $t_1<t\leqslant t_2$ the function $\hat{\zeta}=\hat{\zeta}(t)$ is implicitly represented by[1]

$$-\tfrac{2}{3}(-\hat{\zeta})^{3/2}=2R(t)+\tau\arctan(2\tau R(t),2t-\tau^2)+\arctan(-2R(t),2t-1). \tag{6.7}$$

We get [5] the following final result.

Theorem 17. *Let a_k, $k=1,2,\ldots,$ be the zeros of the Airy function* $\mathrm{Ai}(x)$ *and set*

$$\hat{\zeta}_k=\frac{2^{4/3}a_{n-k+1}}{\nu^{2/3}},\quad k=1,2,\ldots,n. \tag{6.8}$$

Substituting $\hat{\zeta}=\hat{\zeta}_k$ in (5.11) *and inverting with respect to t we obtain $\hat{t}_{n,k}$, which yields a first approximation for the zeros $\lambda_{n,k}^{(\alpha)}$ of $L_n^{(\alpha)}(x)$*

$$\lambda_{n,k}^{(\alpha)}=\nu\hat{t}_{n,k}+\mathrm{O}(\nu^{-1}),\quad k=1,2,\ldots\ . \tag{6.9}$$

Further, the more general and improved formula holds

$$\lambda_{n,k}^{(\alpha)}=\nu\hat{t}_{n,k}+\frac{\hat{t}_{n,k}}{6\nu R(\hat{t}_{n,k})}\left[\frac{5}{(-\hat{\zeta}_k)^{3/2}}+\frac{8\hat{t}_{n,k}^3-12(2-\tau^2)\hat{t}_{n,k}^2+6\hat{t}_{n,k}+\tau^2(1-2\tau^2)}{R^3(\hat{t}_{n,k})(1-\tau^2)}\right]+\mathrm{O}(\nu^{-3}). \tag{6.10}$$

[1] For real x, y, the two-argument function $\arctan(y,x)$, computes the principal value of the argument of the complex number $x+\mathrm{i}y$, so $-\pi<\arctan(y,x)\leqslant\pi$. This function is extended to complex arguments by the formula $\arctan(y,x)=-\mathrm{i}\log((x+\mathrm{i}y)/\sqrt{x^2+y^2})$.

Table 6
First Airy-type approximation

k	$\alpha=1$	$\alpha=10$	$\alpha=100$	$\alpha=1000$	k	$\alpha=1$	$\alpha=10$	$\alpha=100$	$\alpha=1000$
1	1.69	2.24	2.82	3.36	6	3.14	3.46	3.95	4.44
2	2.24	2.72	3.27	3.80	7	3.28	3.57	4.05	4.53
3	2.57	2.99	3.52	4.04	8	3.41	3.68	4.14	4.61
4	2.80	3.18	3.70	4.21	9	3.53	3.78	4.22	4.69
5	2.99	3.33	3.83	4.33	10	3.65	3.88	4.31	4.77

Table 7
Second Airy-type approximation

k	$\alpha=1$	$\alpha=10$	$\alpha=100$	$\alpha=1000$	k	$\alpha=1$	$\alpha=10$	$\alpha=100$	$\alpha=1000$
1	2.91	3.36	3.91	4.45	6	5.89	6.24	6.64	7.12
2	4.11	4.52	5.02	5.55	7	6.13	6.46	6.85	7.32
3	4.79	5.17	5.65	6.15	8	6.34	6.66	7.03	7.49
4	5.26	5.63	6.07	6.57	9	6.53	6.83	7.19	7.64
5	5.61	5.97	6.39	6.88	10	6.71	7.00	7.35	7.79

The bounds for the error terms in (6.9) *and* (6.10) *hold, as* $v \to \infty$, *uniformly with respect to* $k = [pn], [pn]+1, \ldots, n-1, n$ *where* p *is a fixed positive number in* (0.1).

Tables 6 and 7 show the accuracies associated to the approximations obtained, in evaluating the zeros by using (6.9) and (6.10), without the error terms, in the same cases considered in Tables 4 and 5.

The analysis of the results obtained applying Theorem 6.2, suggest to conjecture that the first and the second Airy-type approximations are respectively lower and upper bounds for all the zeros of $L_n^{(\alpha)}(x)$.

The differences in the approximations furnished by the various formulas given in Section 5 and in this section appear more clearly in Fig. 2. It contains the plots of the polygons ρ_{DB}, ρ_{DA}, ρ_{DB^*}, ρ_{DA^*}, ρ_T, and ρ_S, whose vertices are the accuracies corresponding respectively to (6.4), (6.5), (6.9), (6.10), (5.13) and (5.18), in the case $n = 20$ $\alpha = 100$.

7. Simplified nonuniform approximations

Should we be interested in obtaining simple approximations for the first or the last few zeros, or for the zeros in the middle of the oscillatory interval $(0, v)$, it is convenient to transform the previous uniform results in nonuniform ones.

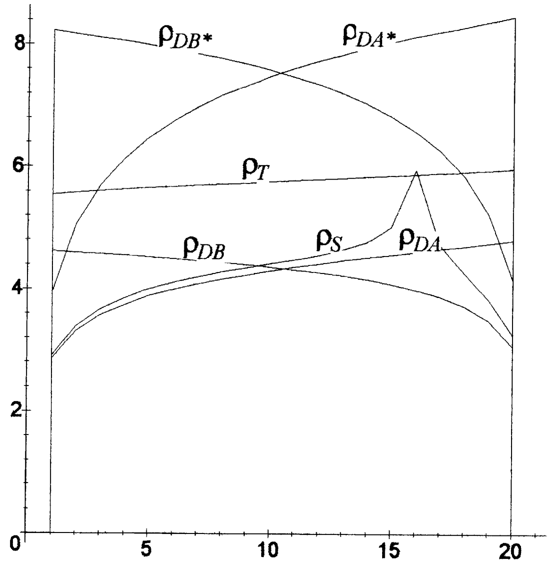

Fig. 2. The accuracies for $n = 20$ and $\alpha = 100$.

By using the same notations of Section 6, let us expand the right-hand side of (6.2) in terms of the ratio

$$\frac{t-t_1}{t_2-t_1} = \frac{t-t_1}{\sqrt{1-\tau^2}}.$$

We get, for $0 \leqslant t - t_1 \leqslant c/v$ with fixed c,

$$\begin{aligned}
&\sqrt{\zeta-4\tau^2} - 2\tau \arctan\frac{\sqrt{\zeta-4\tau^2}}{2\tau} \\
&= c_3\left(\frac{t-t_1}{t_2-t_1}\right)^{3/2} + c_5\left(\frac{t-t_1}{t_2-t_1}\right)^{5/2} + \mathrm{O}((t-t_1)^{7/2}) \\
&= c_3\left(\frac{t-t_1}{\sqrt{1-\tau^2}}\right)^{3/2}\left[1 + \frac{2}{3}\frac{c_5}{c_3}\frac{t-t_1}{\sqrt{1-\tau^2}} + \mathrm{O}\big((t-t_1)^2\big)\right]^{3/2},
\end{aligned} \tag{7.1}$$

where

$$c_3 = \frac{8}{3}\frac{1-\tau^2}{1-\sqrt{1-\tau^2}}, \tag{7.2}$$

$$c_5 = -\frac{4}{5}\frac{(1-\tau^2)(3\sqrt{1-\tau^2}+1)}{(1-\sqrt{1-\tau^2})^2}. \tag{7.3}$$

Analogously, for $\zeta - 4\tau^2 = \mathrm{O}(v^{-1})$, we have

$$\sqrt{\zeta-4\tau^2} - 2\tau \arctan \frac{\sqrt{\zeta-4\tau^2}}{2\tau}$$
$$= \frac{1}{12}\frac{(\zeta-4\tau^2)^{3/2}}{\tau^2} - \frac{1}{80}\frac{(\zeta-4\tau^2)^{5/2}}{\tau^4} + \mathrm{O}((\zeta-4\tau^2)^{7/2})$$
$$= \frac{1}{12}\frac{(\zeta-4\tau^2)^{3/2}}{\tau^2}\left[1 - \frac{1}{10}\frac{\zeta-4\tau^2}{\tau^2} + \mathrm{O}((\zeta-4\tau^2)^2)\right]^{3/2}, \tag{7.4}$$

which, together with (7.1), yields

$$\frac{t-t_1}{\sqrt{1-\tau^2}} + \frac{2}{3}\frac{c_5}{c_3}\left(\frac{t-t_1}{\sqrt{1-\tau^2}}\right)^2 = (12c_3\tau^2)^{-2/3}\left[\zeta - 4\tau^2 - \frac{(\zeta-\tau^2)^2}{10\tau^2} + \mathrm{O}(v^{-3})\right],$$

for $0 \leqslant |\zeta - \tau^2| \leqslant c/v$ with fixed c.

Inverting with respect to $t - t_1$ we get

$$t = t_1 + \frac{\sqrt{1-\tau^2}}{2^{4/3}3^{2/3}\tau^{4/3}c_3^{2/3}}(\zeta-4\tau^2)\left[1 - \left(\frac{1}{10}\frac{1}{\tau^2} + \frac{c_5}{2^{1/3}3^{5/3}\tau^{4/3}c_3^{5/3}}\right)(\zeta-4\tau^2) + \mathrm{O}(v^{-2})\right].$$

Thus, we can state the following theorem which furnishes approximate values of the smallest zeros of $L_n^{(\alpha)}(x)$ in terms of the zeros of $J_\alpha(x)$.

Theorem 18. *Let*

$$\zeta_{n,k} = \left(\frac{4j_{\alpha,k}}{v}\right)^2, \quad k = 1, 2, \ldots, k_0,$$

being $k_0 < n$ a fixed positive integer. Then, as $v \to \infty$, we have

$$\lambda_{n,k}^{(\alpha)} = \frac{v}{2}(1 - \sqrt{1-\tau^2}) + vC_1(\zeta_{n,k} - 4\tau^2)[1 + C_2(\zeta_{n,k} - 4\tau^2) + \mathrm{O}(v^{-2})], \tag{7.5}$$

where

$$C_1 = \frac{(1-\sqrt{1-\tau^2})^{2/3}}{2^{10/3}\tau^{4/3}(1-\tau^2)^{1/6}}, \tag{7.6}$$

$$C_2 = \frac{1}{10\tau^2}\left[\frac{\tau^{2/3}(1+3\sqrt{1-\tau^2})}{2^{7/3}(1-\tau^2)^{2/3}(1-\sqrt{1-\tau^2})^{1/3}} - 1\right]. \tag{7.7}$$

Corollary 19. *Let n be fixed and let*

$$\zeta_{n,k} = \left(\frac{4j_{\alpha,k}}{v}\right)^2, \quad k = 1, 2, \ldots, n.$$

Then, as $\alpha \to \infty$, we have

$$\lambda_{n,k}^{(\alpha)} = \frac{v}{2}(1 - \sqrt{1-\tau^2}) + vC_1(\zeta_{n,k} - 4\tau^2)[1 + C_2(\zeta_{n,k} - 4\tau^2) + \mathrm{O}(\alpha^{-2})], \tag{7.8}$$

where C_1 and C_2 are defined by (7.6) *and* (7.7).

Expand now the right-hand side of (6.7) in terms of the ratio

$$\frac{t_2-t}{t_2-t_1}=\frac{t_2-t}{\sqrt{1-\tau^2}}.$$

We have, for $0\leqslant t_2-t\leqslant c/v$ with fixed c,

$$\begin{aligned}\frac{2}{3}\hat{\zeta}^{3/2}&=\hat{c}_3\left(\frac{t_2-t}{t_2-t_1}\right)^{3/2}+\hat{c}_5\left(\frac{t_2-t}{t_2-t_1}\right)^{5/2}+\mathrm{O}((t_2-t)^{7/2})\\&=\hat{c}_3\left(\frac{t_2-t}{\sqrt{1-\tau^2}}\right)^{3/2}\left[1+\frac{2}{3}\frac{\hat{c}_5}{\hat{c}_3}\frac{t_2-t}{\sqrt{1-\tau^2}}+\mathrm{O}((t_2-t)^2)\right]^{3/2},\end{aligned}\tag{7.9}$$

where

$$\hat{c}_3=\frac{8}{3}\frac{1-\tau^2}{\sqrt{1-\tau^2}+1},\tag{7.10}$$

$$\hat{c}_5=-\frac{4}{5}\frac{(1-3\sqrt{1-\tau^2})(1-\tau^2)}{(\sqrt{1-\tau^2}+1)^2}.\tag{7.11}$$

Then, inverting with respect to t_2-t we get

$$t_2-t=\frac{\sqrt{1-\tau^2}}{(-\hat{c}_3)^{2/3}}\left(-\frac{2^{2/3}}{3^{2/3}}\hat{\zeta}\right)\left[1-\frac{2}{3}\frac{\hat{c}_5}{(-\hat{c}_3)^{2/3}}\left(-\frac{2^{2/3}}{3^{2/3}}\hat{\zeta}\right)+\mathrm{O}(\hat{\zeta}^2)\right].$$

It follows, taking into account of Theorem 6.2 the following expansion for the large zeros of Laguerre polynomials:

Theorem 20. *Let a_m, $m=1,2,\ldots$, be the zeros of the Airy function* $\mathrm{Ai}(x)$. *Then, as $v\to\infty$ and for $m=1,2,\ldots,m_0$ with m_0 fixed, we have*

$$\lambda_{n,n-m+1}^{(\alpha)}=\frac{v}{2}(1+\sqrt{1-\tau^2})+\hat{C}_1a_mv^{1/3}\left[1+\frac{\hat{C}_2}{v^{1/3}}+\mathrm{O}(v^{-4/3})\right],\tag{7.12}$$

where

$$\hat{C}_1=\frac{(\sqrt{1-\tau^2}+1)^{2/3}}{(1-\tau^2)^{1/6}},\tag{7.13}$$

$$\hat{C}_2=\frac{1-3\sqrt{1-\tau^2}}{5(1-\tau^2)^{2/3}(\sqrt{1-\tau^2}+1)^{1/3}}.\tag{7.14}$$

Corollary 21. *Let n be fixed and let a_m, $m=1,2,\ldots,n$ be the first n zeros of the Airy function* $\mathrm{Ai}(x)$. *Then, as $\alpha\to\infty$ and for $m=1,2,\ldots,m_0$ with m_0 fixed, we have*

$$\lambda_{n,n-m+1}^{(\alpha)}=\frac{v}{2}(1+\sqrt{1-\tau^2})+\hat{C}_1a_mv^{1/3}\left[1+\frac{\hat{C}_2}{v^{1/3}}+\mathrm{O}(\alpha^{-4/3})\right],\tag{7.15}$$

where $\hat{C}_1$ and $\hat{C}_2$ are defined by (7.13) *and* (7.14).

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